Differential Equations (Calculus)

Slope Fields

A slope field for $\frac{dy}{dx} = F(x, y)$ consists of short line segments with slope $F(x, y)$ at several points (x, y). A solution curve follows the direction of the line segments.

Euler's Method

For the differential equation $\frac{dy}{dx} = F(x, y)$, if $x_n = x_{n-1} + h$ (where h is the step size) and $y(x_0) = y_0$, then

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

Separation of Variables

To solve the separable differential equation $\frac{dy}{dx} = \frac{g(x)}{h(y)}$, move all terms with y to the left and all terms with x to the right:

$$\int h(y)\,dy = \int g(x)\,dx$$

Orthogonal Trajectories

An orthogonal trajectory intersects a family of curves orthogonally (at right angles), so two mutually orthogonal curves have slopes that are negative reciprocals of each other.

Mixing Problems

If y is the mass of a solute in a tank and t is time, then the rate of change of the mass is

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

Exponential Growth and Decay

Let $k > 0$ and $y(0) = y_0$.

- Exponential growth:
$$\frac{dy}{dt} = ky \qquad y = y_0 e^{kt} \qquad t_2 = \frac{\ln 2}{k}$$
- Exponential decay:
$$\frac{dy}{dt} = -ky \qquad y = y_0 e^{-kt} \qquad t_{1/2} = \frac{\ln 2}{k}$$

Compound Interest

If a principal balance of P dollars grows at an interest rate r and is compounded n times per time period, then after t time periods the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

As $n \to \infty$ (continuous compounding),

$$A = Pe^{rt}$$

First-Order Chemical Reactions

If a chemical reactant A is in a first-order reaction, then the concentration satisfies

$$\frac{d[\mathrm{A}]}{dt} = -k[\mathrm{A}] \qquad [\mathrm{A}] = [\mathrm{A}]_0 e^{-kt}$$

where $[\mathrm{A}]_0$ is the initial concentration, $k > 0$ is the rate constant, and t is time.

Newton's Law of Cooling

If a hot object with an initial temperature T_0 cools down in a surrounding environment of temperature T_s, then the temperature T varies with time t according to

$$\frac{dT}{dt} = -k(T - T_s) \qquad T = T_s + (T_0 - T_s)e^{-kt}$$

where $k > 0$.

Discharging a Resistor–Capacitor Circuit

For a circuit with resistance R and capacitance C, where the time constant is $\tau = RC$, the capacitor's charge Q satisfies

$$\frac{dQ}{dt} = -\frac{1}{\tau}Q \qquad Q = Q_0 e^{-t/\tau}$$

where t is time and Q_0 is the initial charge.

Logistic Growth and Decay

Let $k > 0$ and $y(0) = y_0$.

- Logistic growth:
$$\frac{dy}{dx} = ky\left(1 - \frac{y}{L}\right) \qquad y = \frac{L}{1 + Ce^{-kx}}$$
- Logistic decay:
$$\frac{dy}{dx} = -ky\left(1 - \frac{y}{L}\right) \qquad y = \frac{L}{1 + Ce^{kx}}$$

where $L > 0$ is the carrying capacity (with $0 < y_0 < L$) and C is a constant.

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Properties of Logistic Functions

Let $L > 0$, $C > 0$, and $k > 0$.

Condition	**Logistic Growth**, $\dfrac{L}{1+Ce^{-kx}}$	**Logistic Decay**, $\dfrac{L}{1+Ce^{kx}}$
As $x \to \infty$,	approaches L.	approaches 0.
As $x \to -\infty$,	approaches 0.	approaches L.
At $y = \frac{L}{2}$ (inflection point),	increases most rapidly.	decreases most rapidly.
For $y < \frac{L}{2}$	is concave up.	is concave up.
For $y > \frac{L}{2}$	is concave down.	is concave down.

First-Order Linear Differential Equations

For $y' + P(x)y = Q(x)$, multiply both sides by $\mu(x) = e^{\int P(x)\,\mathrm{d}x}$ and recognize the left side as the derivative of the product $ye^{\int P(x)\,\mathrm{d}x}$. The general solution is

$$ye^{\int P(x)\,\mathrm{d}x} = C + \int Q(x)e^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x$$